Prime ends

$\not\in$ Ω - bounded domain. (Remark: the same in spherical metric).

Lemma (Koebe) $f: \mathbb{D} \to \Omega$ - conformal, γ - crosscut in $\Omega \Rightarrow f^{-1}(\gamma)$ is crosscut in $\mathbb{D}$.

Pf $f^{-1}(\gamma)$ - an arc in $\mathbb{D}$. Need $\lim_{t \to 0,1} f^{-1}(\gamma(t))$

Assume $\lim_{t \to 0} f^{-1}(\gamma(t))$ does not exist

$\Rightarrow \exists (t_n)(t_n') \to 0$, $\gamma(t_n) \to z_0 \neq z_1 \leftarrow \gamma(t_n')$

By Lavrentiev (or extended version) $\lim_{n \to \infty} \operatorname{diam} \gamma(t_n, t_n') \geq c|z_1 - z_2|^2$.

$\operatorname{diam} \gamma(t_n, t_n') \to 0$. contradiction ∎

Def Chain $(\gamma_n, \mathcal{D}_n)$ in Ω: γ_n - crosscut in Ω, $\mathcal{D}_n$ - one of the components of $\Omega \setminus \gamma_n$, such that.

1) $\operatorname{diam} \gamma_n \to 0$ 2) $\mathcal{D}_n \supset \mathcal{D}_{n+1}$ 3) $\operatorname{dist}(\gamma_n, \gamma_{n+1}) > 0$.


Yes

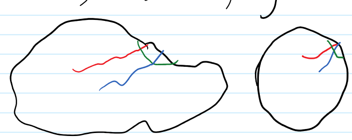

No

Def Two chains are equivalent, $(\gamma_n, \mathcal{D}_n) \sim (\gamma_n', \mathcal{D}_n')$ if $\forall n$ $\mathcal{D}_n$ contains all but finitely many $\mathcal{D}_m'$, and the same is true for $\mathcal{D}_n'$ and $\mathcal{D}_m$.

Def. Prime end - equivalency class of crosscuts.

Lemma. Preimage of a chain under $f: \mathbb{D} \to \Omega$ is a chain.

Pf. $f^{-1}(\gamma_n)$ - crosscut, $\operatorname{diam} f^{-1}(\gamma_n) \leq c\sqrt{\operatorname{diam} \gamma_n}$ - by our remark, $\operatorname{dist} > 0$, By Wolff, $f^{-1}(\mathcal{D}_{n+1}) \subset f^{-1}(\mathcal{D}_n)$ ∎

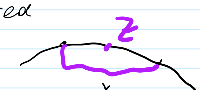


Corollary. f^{-1}(prime end) - prime end.

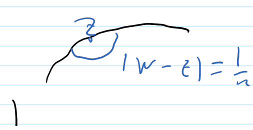
Def. P - prime end, support of prime end $I(P) := \bigcap \overline{\mathcal{D}_n}$ (does not depend on chain).

$I(P)$ - compact, connected

Examples: 0) $\mathbb{D}$



$I(P) = \{z\} \in \mathbb{T}$



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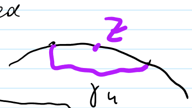
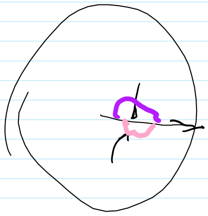


$I(P) = \{z \in \mathbb{T}$
$(\operatorname{dist}(\gamma_n(0), \gamma_n(1)) \to 0)\}$

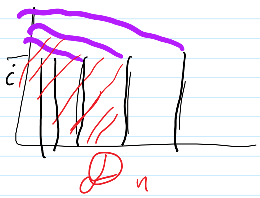
$\Omega = \mathbb{D} \setminus [0,1]$

1)



Two prime ends for which
$I(P) = \{z\}$ if $z \in [0,1]$.

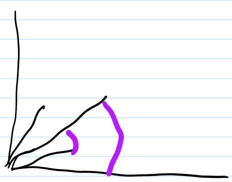
2)



$\Omega = \{\operatorname{Re} z > 0, \operatorname{Im} z > 0\} \setminus \bigcup_{j=1}^{\infty} \{\operatorname{Re} z = \frac{1}{j}, 0 \le \operatorname{Im} z \le 1\}$.

$I(P) = [0, i]$.

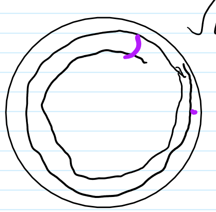
3)



$\Omega = \{\operatorname{Re} z > 0, \operatorname{Im} z > 0\} \setminus \bigcup_n \bigcup_{k \le 2^{n-1}} \{\arg z = \frac{2k-1}{2^n}, 0 \le |z| \le \frac{1}{2^n}\}$.

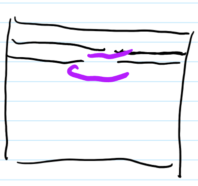
Uncountably many P with
$I(P) = \{0\}$.

4)



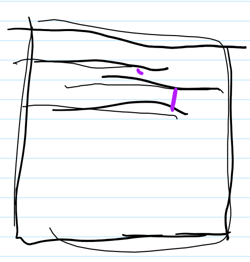
$\Omega = \mathbb{D} \setminus \gamma$, γ – spiral over $\mathbb{T}$.
Then $\exists P$: $I(P) = \mathbb{T}$.

5)



$I(P) = [0,1]$

6)



$I(P) = [0,1]$.

Thm (Caratheodory) $f: \mathbb{D} \to \Omega$. The correspondence $P \to \bigcap f^{-1}(P_n)$ is a bijection

Thm (Caratheodory) $f:\mathbb{D}\to\Omega$. The correspondence $p \to \bigcap \overline{f^{-1}(\mathcal{D}_n)}$ is a bijection between prime ends and $\partial\mathbb{D}$.

Pf. 1) $f^{-1}(P)$ – prime end $\Rightarrow$ $\forall P\ \exists!$ point in $\mathbb{T}$, $\bigcap \overline{f^{-1}(\mathcal{D}_n)}$.

2) Injective: Let $P\to\xi$, $P'\to\xi \Rightarrow (f^{-1}(\gamma_n), f^{-1}(\mathcal{D}_n)) \sim (f^{-1}(\gamma_n'), f^{-1}(\mathcal{D}_n')) \overset{\text{take } f}{\Rightarrow}$ $(\gamma_n, \mathcal{D}_n) \sim (\gamma_n', \mathcal{D}_n')$.

3) Surjective: Take $\xi\in\partial\mathbb{D}$. Take $z_n\in\mathbb{D}$, $(z_n)\to\xi$.
By Wolff, $\exists$ crosscut γ_n such that $f^{-1}(\gamma_n)$ separates ξ and z_n from 0, $\operatorname{diam}\gamma_n \le \frac{C}{\sqrt{\log \frac{1}{|z_n-\xi|}}}$, and $f^{-1}(\gamma_n)$ – circular arc of radius r_n, centered at ξ

By Lavrentiev, $r_n\to 0$. Select $r_{n_k}\downarrow 0$

Then $(\gamma_{n_k}, f(\{|z|<1, |z-\xi|<r_{n_k}\}))$ – prime end. ∎

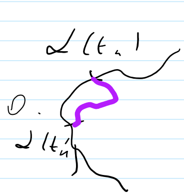
Corollary. $(\gamma_n, \mathcal{D}_n) \not\sim (\gamma_n', \mathcal{D}_n') \Leftrightarrow \exists n: \mathcal{D}_n\cap\mathcal{D}_n' = \emptyset$.
Pf. Pull back to $\mathbb{D}$ ∎

Def. $\mathcal{P}(\Omega)$ – set of prime ends – Caratheodory boundary. $\hat\Omega := \Omega \cup \mathcal{P}(\Omega)$, $\hat f:\mathbb{D}\to\hat\Omega$.
Caratheodory metric extends to $\mathcal{P}(\Omega)$: the shortest crosscut separating
As before, $c|z_1-z_2|' \le \rho(\hat f(z_1), \hat f(z_2)) \le \frac{C}{\sqrt{\log\frac{1}{|z_1-z_2|}}}$. $z_1\in\partial\Omega$, $z_2\in\Omega$ from w_0

($\partial\Omega$ – Jordan curve)

Thm (Caratheodory). Ω – Jordan domain, then conformal $f:\mathbb{D}\to\Omega$ can be extended to homeo: $\hat f:\overline{\mathbb{D}}\to\overline{\Omega}$.

Pf. Need to check:
1) $I(P)$ – single point $\forall P$: γ_n joins $\alpha(t_n)$ to $\alpha(t_n')$, α – homeo $\Rightarrow$ $|t_n - t_n'|\to 0$.
$|\alpha(t_n)-\alpha(t_n')|\to 0$
so $\lim\alpha(t_n) = \lim\alpha(t_n') = I(P)$.



2) Every point in $I(P)$ at the boundary. : take $\xi\in\partial\Omega$. $\mathcal{D}_n$ = component $B(\xi,\frac{1}{n})\cap\Omega$ containing ξ
$\bigcap\mathcal{D}_n = \{\xi\}$
$\gamma_n := \partial\mathcal{D}_n \setminus \partial\Omega$

Remark. Same way can prove: $f: \mathbb{D} \to \Omega$ extends to $f: \overline{\mathbb{D}} \to \overline{\Omega}$ iff $\partial\Omega$ is locally connected, i.e. $\forall \varepsilon > 0 \ \exists \delta > 0: z_1, z_2 \in \partial\Omega, |z_1 - z_2| < \delta \Rightarrow \exists C \subset \partial\Omega$ $z_1, z_2 \in C$, C – connected, $\operatorname{diam} C < \varepsilon$.